Shivaji University, Kolhapur

Question Bank For Mar 2022 (Summer) Examination

M.Sc. I Sem. I (Statistics/Applied Statistics and Informatics) Exam

Subject Code: **83439**
Subject Name: **Linear Algebra**

Short answer questions (2 marks)

1 Define vector space with an example.
2 Define spanning set. Give an example.
3 Define basis of a vector space. Give an example.
4 Define basis and orthonormal basis.
5 Define linearly independent set of vectors. Is the set {(1, 2), (2, 2), (2, 3)} linearly dependent?
6 State about the linear dependency of the following sets.
A={}
B={}
7 Examine whether the vectors **a** = (2, 7,−8), **b** = (5, 1,−4), and **c** = (3, 5,−4) are linearly independent.
8 Define dimension of a vector space. What is the dimension of $\mathbf{V}_3 = \{(x, x, y): x, y \in (-\infty, \infty)\}$?
9 Define null space and nullity.
10 Define symmetric and skew-symmetric matrices.
11 Define Hermitian and skew hermitian matrix.
12 What is rank of a matrix? State any two properties of it.
13 Define permutation matrix. Give one example.
14 Define reducible matrix. Give one example.
15 Define G-inverse. Obtain a g-inverse of [1 2 3].
16 Define Moore-Penrose inverse.
17 Determine any two g-inverses of **a** = [1, 3, 5].
18 If matrix **G** is a g-inverse of a matrix **A**, show that **H** = **GAG** is a g-inverse of **A** such that rank (**H**) = rank (**A**).
19 Prove that inverse of a square matrix exists if and only if the matrix is nonsingular.
20 Prove that if inverse of a square matrix exists, it is unique.
21 Prove that all nonsingular matrices of the same order have the same rank.
22 Define homogeneous system. State a necessary and sufficient for existence of its non-trivial solution.
23 State necessary and sufficient conditions for
  a) a system of linear equations Ax = b to be consistent.
  b) a system of linear equations Ax = b to have a unique solution.
24 What is the maximum number of linearly independent solutions to a system of linear equations **Ax** = **0**, where **A** is a 3×5 matrix of rank 3?
25 State a necessary and sufficient condition for a system of equations **Ax** = **b** to be inconsistent.
26 What is characteristic value problem?
27 What are the eigen values of a 2×2 matrix whose determinant is 12 and trace is 8?
28 Define algebraic and geometric multiplicities of a characteristic root of a matrix.
29 The eigen values of a 2×2 matrix are 2 and 6. What are its determinant and trace?
30 State how the singularity of a square matrix is related to its eigen values.
31 Prove that an idempotent matrix is singular if and only if zero is its eigen value.
32 If λ is an eigen value of a nonsingular matrix **A**, show that $1/\lambda$ is an eigen value of $\mathbf{A}^{-1}$.

33 Prove that the eigen values of an idempotent matrix are either 0 or 1.
34 Define an idempotent matrix. State its eigen values.
35 Define an orthogonal matrix. State its eigen values.
36 Define quadratic form. State the type of the quadratic form $x_1^2 + x_2^2$?
37 Define signature and index of quadratic form.
38 State the matrix associated with the quadratic form $(x_1 + 5x_2 - 3x3)^2$ and $x_1^2 - 2x_3^2 - x_1x_2$.
39 State a necessary and sufficient condition for a real quadratic form to be positive definite.
40 State the type of the quadratic forms $x_1^2 + x_2^2$ and $x_1^2 - 2x_3^2 - x_1x_2$.

Long answer questions (8 marks)

1 Prove that a set of vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_k\}$is linearly dependent if and only if someone of the $\boldsymbol{a}_i$'s can be expressed as a linear combination of the rest.
2 Show that every basis for n-dimensional Euclidean space contains exactly n vectors.
3 Show that any subset of size (n – 1) of the set of vectors n vectors $\{ \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n \}$ in n-dimensional Euclidean space is linearly independent, where

$\boldsymbol{x}_1 = (1, -1, 0, 0, \ldots, 0)$,
$\boldsymbol{x}_2 = (1, 0, -1, 0, \ldots, 0)$,
$\boldsymbol{x}_3 = (1, 0, 0, -1, \ldots, 0)$,
$\vdots$
$\boldsymbol{x}_{n-1} = (1, 0, 0, \ldots, -1)$, and
$\boldsymbol{x}_n = (n - 1, -1, -1\ldots -1)$.

4 Define orthogonal and orthonormal basis. Describe Gram-Schmidt orthogonalisation.
5 Given a basis $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\}$ for *n*-dimensional space and a non-null vector $\boldsymbol{b}$ in *n*-dimensional space, show that if any vector $\boldsymbol{a}_i$ for which $\alpha_i \neq 0$ in the representation of $\boldsymbol{b}$ as $\boldsymbol{b} = \sum_{i=1}^{n} \alpha_i \boldsymbol{a}_i$ is replaced from$\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\}$ by $\boldsymbol{b}$, then the new set is also a basis for *n*-dimensional space.
6 Show that any set of *n* linearly independent vectors in *n*-dimensional space forms a basis for *n*-dimensional Euclidean space.
7 Define and illustrate giving one example each: i) Linear Independence ii) Basis. Prove that the representation of unit vectors in terms of a basis vector is unique.
8 Prove that the n unit vectors $e_1, e_2, \ldots, e_n$ form a basis and any set of n mutually orthogonal vectors form a basis.
9 Prove that any subset of linearly independent set of vectors is linearly independent and any superset of linearly dependent vectors is linearly dependent.
10 Define symmetric and skew symmetric matrix. Give one example each. If S is real symmetric matrix, prove that $(I + S)^{-1}(I - S)$ is an orthogonal matrix.

11 Define rank of matrix. Let **A** and **B** be $m\times n$ and $n\times p$ matrices, respectively. Show that rank (**AB**) $\leq$ min {rank(**A**), rank(**B**)}.
12 Define an orthogonal matrix. Show that the columns of an orthogonal matrix are linearly independent and converse of this is not true.
13 Define determinant of a $n \times n$ matrix, $n \geq 1$. Consider a matrix $a = (a_{ij})$ where $a_{ij} = a$ if $i = j = 1, \ldots, n$ and $a_{ij} = b$ if $i \neq j = 1, \ldots, n$.
Show that $\det(A) = [a + (n-1)b](a-b)^{n-1}$
14 Show that the rank of sum of two matrices cannot exceed sum of their ranks.
15 Explain the computation of the inverse of higher order matrix by partitioning.
16 Prove that every matrix has g-inverse. Also prove that matrix **G** is a g-inverse of matrix **A** if and only if **AGA** =**A**.
17 Define g-inverse and illustrate giving two examples. Prove that g-inverse always exist and it is not unique.

18 Show that the maximum number of linearly independent solutions to the system of equations $\mathbf{Ax} = \mathbf{0}$ is n-rank($\mathbf{A}$), where n is the number of columns in $\mathbf{A}$.

19 Show that the system of equations $AX = b$ is consistent if and only if $\rho(A) = \rho(A\colon b)$

20 Define Moore - Penrose (MP) inverse. Show that each m×n matrix A, there exists one and only one n×m matrix A^{+} (MP-inverse) with stating conditions.

21 Define characteristic roots and vectors and show that if A is a real symmetric matrix then all characteristic roots and vectors are real.

22 Define characteristic root and find the same for idempotent and orthogonal matrix. Show that characteristic vectors corresponding to different characteristic roots are orthogonal.

23 Explain Algebraic and geometric multiplicities and show that Algebraic multiplicity is grater that geometric multiplicity.

24 Let $\lambda_1, \lambda_2, \ldots, \lambda_n$, be the characteristic roots of an n×n matrix A. Show that $|\mathbf{A}| = \prod_{i=1}^{n} \lambda_i$ and trace (A) $= \sum_{i=1}^{n} \lambda_i$.

25 Show that if a real symmetric matrix **A** has eigen values 0 and 1 only then **A** is idempotent.

26 Show that. i) Every orthogonal matrix A can be expressed as $(I+S)(I-S)^{-1}$ provided -1 is not a characteristic root of A and suitable choice of real skew-symmetric matrix S. ii) If A and B are two square matrices then matrix AB and BA have the same characteristic roots.

27 State and prove the Cayley-Hamilton theorem and explain its application.

28 Find $\mathbf{A}^{-1}$ and $\mathbf{A}^{5}$ using Caley-Hamilton theorem, where $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$ **;** $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -2 & 4 \end{bmatrix}$ **;** $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}$

29 Define symmetric, skew symmetric and orthogonal matrix with example. Show that if S is real skew symmetric matrix then $(I-S)$ is non singular and the matrix $A = (I+S)(I-S)^{-1}$ is orthogonal.

30 Find the characteristic roots and characteristic vectors of $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

31 Define g-inverse and obtain g-inverse of $\begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 2 \\ 2 & 0 & 4 \end{pmatrix}$ and verify the same.

32 Define g-inverse and Moore-Penrose generalized inverse. Obtain MP inverse of $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}$.

33 Find the inverse of the partitioned matrix.

M= $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where,

A= $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, B= $\begin{bmatrix} 0 & 2 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$, C= $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \end{bmatrix}$, D= $\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

34 Find the inverse of the partitioned matrix.

$M=\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where,

$A=\begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix}$, $B=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C=\begin{bmatrix} 1 & 1 \end{bmatrix}$, $D=[4]$

35 Find inverse of the following matrices by partitioning method.

$M=\begin{bmatrix} 2 & 1 & 0 \\ 3 & 0 & 5 \\ 7 & 6 & 4 \end{bmatrix}$

36 Find inverse of the following matrices by partitioning method.

$N=\begin{bmatrix} 1 & 0 & 2 & 3 & -1 \\ 1 & 2 & 4 & 4 & 0 \\ 2 & 4 & 0 & -1 & 1 \\ 3 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & 1 & 2 \end{bmatrix}$

37 For what values of 'a', equation X+Y+Z=1, X+2Y+4Z=a, X+4Y+10Z=a^2 has a solution and solve them completely in each cases.

38 Check whether following system of equations are consistent and if to solve the same.

I. $2X_1+X_2+4X_3=16$
$3X_1+2X_2+X_3=10$
$X_1+2X_2+3X_3=16$

II. $2X_1+6X_2=-11$
$6X_1+20X_2-6X_3=-3$
$6X_2-18X_3=-1$

39 Show that quadratic form X^TAX is positive definite if and only if the characteristic roots of A are all positive.

40 Let A be a symmetric matrix with a characteristic root $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Prove that $Sup_X \frac{X^TAX}{X^TX} = \lambda_1$

41 Define quadratic form and discuss definiteness of quadratic form with suitable example.

42 Show that that the definiteness of the quadratic form is invariant under non-singular transformation. Also show that any quadratic form can be transformed to a form containing only square terms.

43 Define symmetric and skew-symmetric matrices. Let A and B be two symmetric matrices such that characteristic roots $|A - \lambda B| = 0$ are all district then show that there exists a matrix P such that P^TAP and P^TBP are both diagonal matrices.

44 Explain extrema of quadratic form. Show that if atleast one of A and B is positive definite or negative definite then X^TAX and X^TBX can be simultaneously reduced to diagonal form by non-singular transformation.

45 State and prove a necessary and sufficient condition for a real quadratic form to be positive definite.

46 Reduce the following quadratic form to a form containing only square terms. $x_1^2 + x_3^2 + 4x_1x_3 + 8x_2x_3$. Examine whether the following quadratic form is positive definite. $x_1^2 + x_2^2 + 2x_3^2 + 2x_2x_3$.

47 Reduce following quadratic forms into forms containing only square terms & classify them. Determine rank of the matrix.

i) $(X_1 + X_2 + X_3)^2$

ii) $X_1^2 + X_2^2 - 3X_3^2 + 2X_1X_2 - 6X_2X_3$

48 Obtain spectral decomposition of **A** = $\begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$ and hence find A^3 and A^4.

49 Explain Spectral decomposition of a real symmetric matrix A. For given matrix A find 5^{th} power of matrices. $\begin{pmatrix} 1 & 0.3 & 0.3 \\ 0.3 & 2 & 0.7 \\ 0.3 & 0.7 & 2 \end{pmatrix}$.

50 Explain :

i) Singular value decomposition,

ii) Choleskey decomposition

Short notes (4 Marks each)

1 Vector space
2 Gram-Schmidt orthogonalisation
3 Elementary operations on matrix
4 Inverse of a matrix
5 Inverse of a partitioned matrix
6 Permutation matrix
7 Kronecker product
8 Generalized inverse
9 Procedures to obtain generalized inverse of a matrix
10 Moore-Penrose generalized inverse
11 System of linear equations
12 Characteristic roots and vectors of a matrix
13 Cayley-Hamilton theorem
14 Spectral decomposition of a real symmetric matrix
15 Singular value decomposition
16 Quadratic form and non-singular transformation
17 Definiteness of quadratic forms
18 Choleskey decomposition
19 Extrema of a quadratic form
20 Simultaneous reduction of two quadratic forms