Shivaji University , Kolhapur

Question Bank for March 2022 (Summer) Examination

Subject Code : **80960** Subject Name : **Real Analysis**

Objective Questions

1) $m^*(\emptyset) =$
   a) 1
   b) $m^*(\{1\})$
   c) 2
   d) $m^*(\mathbb{R})$

2) $m^*[C \cap (\mathbb{R} - C)]$=
   a) 0
   b) 1
   c) $m^*(C)$
   d) $m^*(\mathbb{R} - C)$
3) Outer measure is countably _________
   a) Sub-additive
   b) Additive
   c) Non-additive
   d) Translation variant
4) $m^*(A \cup B) = m^*(B)$ is true only if
   a) $m^*(A) = 0$
   b) $\mathrm{m}^*(\mathrm{B}) = 0$
   c) $m^*(A \cup B) = 0$
   d) $m^*(A \cap B) = 0$
5) $m^*(\mathbb{Q})$=
   a) $m^*(\mathbb{N})$
   b) ∞
   c) 1
   d) 2
6) A set E is said to be measurable, if for any set A,
   a) $m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$
   b) $m^*(E) = m^*(A \cap E) + m^*(A \cap E^c)$
   c) $m^*(E^c) = m^*(A \cap E) + m^*(A \cap E^c)$
   d) $m^*(A) = m^*(A \cap E) \cup m^*(A \cap E^c)$
7) Any countable set is
   a) Measurable
   b) Non-measurable
   c) Open
   d) Closed

8) Finite union of measurable set is
   a) Measurable
   b) Non-measurable
   c) Open
   d) Closed
9) The set of all measurable set is
   a) Measurable
   b) Non-measurable
   c) Algebra
   d) Sigma Algebra
10) For measurable set E we say that a property holds almost everywhere on E if wherever the property fails has measure
   a) Zero
   b) Non-zero
   c) Unity
   d) Infinity
11) If E_1 and E_2 are measurable then
   a) $m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$
   b) $m(E_1 \cup E_2) \cup m(E_1 \cap E_2) = m(E_1) \cup m(E_2)$
   c) $m(E_1 + E_2) + m(E_1 - E_2) = m(E_1) + m(E_2)$
   d) $m(E_1 \cup E_2) - m(E_1 \cap E_2) = m(E_1) - m(E_2)$
12) I) G_δ sets are open
   II) F_σ Sets are open
   a) Both I and II true
   b) Only I is true
   c) Only II is true
   d) Both I and II are false
13) I) Set of all real numbers is measurable
   II) Empty set is not measurable.
   a) Both I and II true
   b) Only I is true
   c) Only II is true
   d) Both I and II are false
14) $m([0,1] \cup [3,4]) =$
   a) 0
   b) 1
   c) 2
   d) 3
15) Cantor set is
   a) Closed and countable
   b) Open and countable
   c) Closed and uncountable
   d) Open and uncountable

16) Difference between any two points in choice set CE is
   a) Rational
   b) Irrational

c) Integer
d) Complex

17) If f is defined on E of measure zero then
a) f is always measurable
b) f may or may not be measurable
c) f is never measurable
d) Measurability of f depends on definition of f

18) If f is a function defined on measurable E then f is measurable if inverse image of each open set under f is
a) Open
b) Closed
c) Measurable
d) Non-measurable

19) If f is a real valued function on E, for measurable subset D of E then f is measurable if and only if the restriction of f to ______ and _______ are measurable
a) E and E-D
b) E and E+D
c) E+D and E-D
d) None

20) If f and g are measurable functions then which of the following are not measurable
a) f.g
b) f+g
c) f/g
d) f-g

21) |f|=
a) $f^{+} + f^{-}$
b) $f^{+} - f^{-}$
c) $f^{+} * f^{-}$
d) f^{+}/f^{-}

22) I) The pointwise limit of continuous function may not be continuous
II) The pointwise limit of Riemann integrable function is Riemann integrable
a) Both I and II are true
b) Only I is true
c) Only II is true
d) Both I and II are false

23) $\{x \in \mathbb{R} | \chi_A(x) \geq c\}$= _________, if $0 \leq c \leq 1$
a) A
b) $\mathbb{R}$
c) ϕ
d) None of these

24) I) Non-measurable functions does not exist
II) Sum of two non-measurable functions is non-measurable
a) Both I and II are true
b) Only I is true
c) Only II is true
d) Both I and II are false

25) For any sets A and B$\chi_{A \cup B} = \chi_A + \chi_B$ is true if
a) $A \cup B = \phi$

b) $A \cap B = \phi$
c) $A = B$
d) $A \subseteq B$

26) For any set A and B $\chi_{A \cup A^c} =$
a) $\chi_{\mathbb{R}}$
b) χ_A
c) χ_{A^c}
d) χ_ϕ

27) Which of the following is not Littlewoods's Principle
a) Every measurable set is nearly finite union of intervals
b) Every measurable function is nearly continuous
c) Every pointwise convergence of sequence is nearly uniform convergence
d) Every continuous function is nearly uniform continuous

28) Let $f(x) = 5\chi_{E_1} + 4\chi_{E_2}$ where E_1=[1,2] and E_2=[3,5] then integral of f is _________
a) 13
b) 9
c) 14
d) 12

29) Let f be bounded real valued function defined on set of finite measure E then the lower Lebesgue integral of f is given by_______________
a) $sup\left\{\int_E \phi \mid \phi \text{ is simple and } \phi \leq f\right\}$
b) $inf\left\{\int_E \phi \mid \phi \text{ is simple and } \phi \leq f\right\}$
c) $sup\left\{\int_E \psi \mid \psi \text{ is simple and } f \leq \psi\right\}$
d) $inf\left\{\int_E \psi \mid \psi \text{ is simple and } f \leq \psi\right\}$

30) Let f be bounded real valued function defined on set of finite measure E and A and B be disjoint measurable subsets of E then which of the following is true
a) $\int_{A \cup B} f = \int_A f + \int_B f$
b) $\int_{A \cup B} f \leq \int_A f + \int_B f$
c) $\int_{A \cup B} f \geq \int_A f + \int_B f$
d) $\int_{A \cup B} f \neq \int_A f + \int_B f$

31) Let f be bounded real valued function defined on set of finite measure E then choose the correct alternative
a) $\left|\int_E f\right| = \int_E |f|$
b) $\left|\int_E f\right| \geq \int_E |f|$
c) $\left|\int_E f\right| \leq \int_E |f|$
d) $\left|\int_E f\right| \neq \int_E |f|$

32) Let f be bounded real valued function defined on set of finite measure E and $A \leq f \leq B$ then choose the correct alternative
a) $A.m(E) \leq \int_E f \leq B.m(E)$
b) $A \leq \int_E f \leq B$

c) $A.m(E) \leq \int_E f \leq B$

d) $A \leq \int_E f \leq B.m(E)$

33) Let f be bounded real valued function defined on set E of measure zero then $\int_E f =$

a) 0
b) 1
c) 3
d) 5

34) Let f be non-negative measurable function on E then for any $\lambda > 0$ choose the correct alternative

a) $m\{x\epsilon E|f(x) \geq \lambda\} \leq \frac{1}{\lambda}\int_E f$

b) $m\{x\epsilon E|f(x) \geq 0\} \leq \frac{1}{\lambda}\int_E f$

c) $m\{x\epsilon E|f(x) \geq \lambda\} \leq \int_E f$

d) $m\{x\epsilon E|f(x) \geq 0\} \leq \int_E f$

35) Let f and g are non-negative measurable function on E then

$\int_E af + bg = a\int_E f + b\int_E g$is true for

a) All real numbers a and b
b) a>0 and b>0
c) a<0 and b<0
d) a>0 and b<0

36) Let f be measurable function then consider

I)f^+, f^- are integrable$\Longrightarrow$ $|f|$ is integrable

II)$|f|$isintegrable$\Longrightarrow$ f^+, f^- areintegrable

a) Both I and II are true
b) Only I is true
c) Only II is true
d) Both I and II are false

37) A measurable function f is said to Lebesgueintegrable over set E if _______ is integrable over E.

a) |f|
b) –f
c) f^2
d) $-f^2$

38) A non-negative measurable function f is said to integrable over a measurable set E if _________

a) $\int_E f < \infty$

b) $\int_E f > 0$

c) $\int_E f < 0$

d) $\int_E f = \infty$

39) If E is the set contained in an open interval $(a, b)$and there is an increasing function on (a, b)that fails to be differentiable at each point in E then …..

a) $m^*(E) = 0$
b) $E$is of infinite measure
c) $E$is non measurable set
d) None of theses

40) If $f$is an increasing function on the closed, bounded interval$[a, b]$ then $f'$is integrable over $[a, b]$ and ….

a) $\int_a^b f' \leq f(b) - f(a)$
b) $\int_a^b f' \leq f(a) - f(b)$
c) $\int_a^b f' \geq f(b) - f(a)$
d) $\int_a^b f' = f(b) - f(a)$

41) If f is an increasing function on the closed, bounded interval$[a, b]$ then f is of bounded variation on $[a, b]$and ….

a) $TV(f) = f(b) - f(a)$
b) $TV(f) < f(b) - f(a)$
c) $TV(f) > f(b) - f(a)$
d) $TV(f) = f(a) - f(b)$

42) If f is Lipschitz on $\mathbb{R}$ and g be absolutely continuous on $[a, b]$ then the composition fog is ….

a) absolutely continuous on $[a, b]$
b) need not be absolutely continuous on $[a, b]$
c) Uniformly continuous
d) Continuous

43) 11) I)l^∞ is normed linear space.

II)$l^1$is not normed linear space.

a) Only I is true
b) Only II is true
c) Both I and II are true
d) Both I and II are false

44) If $p = 1$then its conjugate number $q = \cdots$

a) 1 b) 0 c) ∞ d) -1

45) For $1 < p < \infty$, q the conjugate of $p$and any two positive numbers a and b

a) $ab \le \frac{a^p}{p} + \frac{b^q}{q}$

c) $ab \le \frac{a^p}{a} + \frac{b^q}{b}$

b) $ab \le \frac{a^p}{q} + \frac{b^q}{p}$

d) $ab \le \frac{a^p}{ap} + \frac{b^q}{qb}$

Descriptive Questions

1) Define Lebesgue outer measure and prove that

   i) $m*(A) \le m*(B)$ if $A \subseteq B$

   ii) $m*(A+x) = m*(A)$

2) Give construction of cantor set C and prove that C is measurable.

3) Prove that the interval (a, ∞) is measurable.

4) Define Lebesgue outer measure. Prove that outer measure is countably subadditive and hence show that $m*(A) = 0$ if A is countable.

5) Prove that the collection of all measurable set is a σ - algebra.

6) Prove that Lebesgue measure is invariant under translation.

7) If f is measurable, prove that the set $\{x \mid f(x) = \alpha\}$ is measurable for all $\alpha \in \square$.

8) Prove that a constant function with measurable domain is measurable.

9) If f and g are measurable functions defined on same domain then prove that

   1) $f + c$ 2) $c.f$ 3) $f + g$ 4) $f.g$

are measurable where c is constant.

10) Prove that outer measure of an interval is its length.

11) Prove that for given any set A and $\in > 0$, there is an open set O such that $A \subset O$ and $m^*(O) \leq m^*(A) + \in$. Further there exists a set $G \in G_\delta$ such that $A \subset G$ and $m^*(A) = m^*(G)$.

12) Define measurable set. Prove that if $m^*(E) = 0$ then E is measurable. Further every subset of E is measurable.

13) Let $\{E_n\}$ be countable collection of measurable sets then prove that

$$m\left(\bigcup_n E_n\right) = \lim_{n\to\infty} m\left(\bigcup_{k=1}^{n} E_k\right)$$

Further if $\{E_n\}$ is an increasing sequence of measurable sets then

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m\left(E_n\right).$$

14) Define algebra of a sets and prove that given any collection $\mathcal{C}$ of subsets of X there is a smallest algebra containing $\mathcal{C}$.

15) Let E is measurable set. Then prove that for given $\in > 0$, there is an open set $E \subset O$ such that $m^*(O - E) < \in$. Further prove that there is $G \in G_\delta$ such that $E \subset G$ and $m^*(G - E) = 0$

16) Let E is measurable set. Prove that there exists a Borel set B_1 and B_2 such that $B_1 \subseteq E \subseteq B_2$ and $m\left(B_1\right) = m(E) = m\left(B_2\right)$.

17) Prove that union of finite collection of measurable sets is measurable.

18) Let A be any set and $E_1, E_2, \ldots, E_n$ be a finite sequence of measurable sets then prove that $m * (A \cap [\bigcup_{k=1}^{n} E_k]) = \sum_{k=1}^{n} m * (A \cap E_k)$.

19) Given any set A is measurable and $\in > 0$, there is an open set O such that $A \subset O$ and $m^*(O - A) < \in$.

20) Let E be measurable set of finite outer measurable. Then prove that for each $\epsilon > 0$ there is a finite collection of open intervals $\{I_k\}_{k=1}^{n}$ for which $O = \bigcup_{k=1}^{n} I_k$ such that, $m * (E - O) + m * (O - E) < \epsilon$.

21) E_1 and E_2 are measurable, show that $m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$.

22) Let $\{I_k\}_{k=1}^{\infty}$be a sequence of measurable sets then prove that $m(\bigcup_k E_k) \leq \sum_k E_k$. Also prove if $E_k{}'s$ are pairwise disjoint then $m(\bigcup_k E_k) = \sum_k E_k$.

23) Let $\{A_k\}_{k=1}^{\infty}$be a ascending sequence of measurable sets then prove that $m(\bigcup_{k=1}^{\infty} A_k) = \lim_{k\to\infty} A_k$.

24) Let $\{B_k\}_{k=1}^{\infty}$be a descending sequence of measurable sets and $m(B_1) < \infty$ then prove that $m(\bigcup_{k=1}^{\infty} B_k) = \lim_{k\to\infty} B_k$.

25) State and prove Borel-Cantelli Lemma.

26) Prove that a continuous function defined on measurable set is measurable. Is converse true? Justify with an example.

27) State and prove Egoroff's theorem.

28) Prove that there exists a non-measurable set in $[0,1]$.

29) Prove that a continuous function defined on measurable set is measurable.

30) Let E be a bounded measurable set of real number. Let Λ be a bounded countably infinite set of real numbers for which the collection $\{\lambda + E\}_{\lambda\in\Lambda}$ of translations of E is disjoint then prove that $m(E) = 0$.

31) Prove that any set E of real numbers with positive outer measure contains a subset which is not measurable.

32) Prove that a function f is measurable then the set $\{x|f(x) = c\}$ is measurable for all $c \in \mathbb{R}$.

33) Show that a function defined by, $f(x) = \begin{cases} x+4, & if\ x \geq 2 \\ 8 & if\ x < 2 \end{cases}$ is measurable

34) Let f be a function defined on a measurable set E. Then prove that f is measurable if and only if for each open set O, the inverse image of O under f, $f^{-1}(O)$ is measurable.

35) Let f be an extended real valued function on E, then prove that if f is measurable on E on and $f = g$ a.e. on E, then g is measurable also prove for a measurable subset D of E, f is measurable on E if and only if the restrictions of f to D and $E - D$are measurable.

36) Let f and g be measurable functions on E that are finite a.e. on E then prove that for any α and β, $\alpha f + \beta g$ is measurable and $f.g$ is measurable on E.

37) Let g be a measurable real valued function defined on E and let f be a continuous real valued function defined on $\mathbb{R}$. Then the composition $f \circ g$ is a measurable function on E.

38) Prove that for a finite family $\{f_k\}_{k=1}^{n}$ of measurable functions with common domain E, the functions $\max \{f_1, f_2, \dots, f_n\}$ and $\min \{f_1, f_2, \dots, f_n\}$ are measurable functions.

39) Let $\{f_n\}$ be a sequence of measurable functions on E which converges point wise a.e. on E to a function f.Then prove that f is measurable.

40) Let A be any set. Prove that the characteristics function χ_A of A is measurable if an only if A is measurable.

41) Prove that the sum, product and difference of two simple functions are simple.

42) Let f be a measurable real valued functions on E. Assume that f is bounded on E and there is an integer $M \geq 0$ such that $|f| \leq M$ on E. Then prove that for each $\varepsilon > 0$, there are simple functions ϕ_ε and ψ_ε defined on E such that $\phi_\varepsilon \leq f \leq \psi_\varepsilon$and $0 \leq \psi_\varepsilon - \phi_\varepsilon < \varepsilon$ on E.

43) Prove that an extended real valued function f on a measurable set E is measurable if and only if there is a sequence $\{\phi_n\}$ of simple functions on E which convergesd pointwise on E to f and $|\phi_n| \leq |f|$ for all n, on E.

44) Let E be a measurable set of finite measure. Let $\{f_n\}$ be sequence of measurable functions on Ethat converges pointwise on E to a real valued function f. Then prove that for each $\eta > 0$ and $\delta > 0$, there is a measurable subset A of E and there is an index Nsuch that $|f_n - f| < \eta$ on A for all $n \geq N$and $m(E - A) < \delta$.

45) Let f be a simple function defined on a set E. Then prove that ever $\varepsilon > 0$, there is a continuous function g on $\mathbb{R}$ and a closed set $F \subseteq E$ such that $f = g$ on F and $m(E - F) < \varepsilon$.

46) State and prove Lusin theorem.

47) State and prove bounded convergence theorem.

48) Define Lebesgue integral of non-negative measurable function. Prove that if f and g are non-negative measurable functions, then

i) $\int_E c.f = c\int_E f, \quad c > 0$

ii) $\int_E (f+b) = \int_E f + \int_E g$

49) State and prove Fatous lemma.

50) Define integrable function. Prove that if f and g are integrable function over E then

i) $f \le g \quad \text{a.e} \Rightarrow \int_E f \le \int_E g$

ii) If A and B are disjoint measurable sets contained in E, then

$\int_{A\cup B} f = \int_A f + \int_B f$

51) Prove that f is integrable if and only if $|f|$ is integrable.

52) If f and g are bounded measurable functions defined on set E of finite measure. Prove the following

i) $\int_E (af + bg) = a\int_E f + b\int_E g$

ii) $f = g \quad \text{a.e} \Rightarrow \int_E f = \int_E g$

53) State and prove monotone convergence theorem.

54) Let ϕ and ψ be simple functions which vanishes outside set of finite measure. Then prove that

i) $\int (a\phi + b\psi) = a\int \phi + b\int \psi$

ii) $\phi \ge \psi \quad \text{a.e} \Rightarrow \int \phi \ge \int \psi$

55) Let $\{g_n\}$ be a sequence of integrable functions which converges to an integrable function g a.e. and let $\{f_n\}$ be sequence of measurable functions such that $|f_n| \le g_n$ for all n and $f_n \to f$ a.e. Then prove that if $\int_E g = \lim \int_E g_n$ then $\int_E f = \lim \int_E f_n$.

56) Let $\{E_i\}_{i=1}^{n}$ be a finite disjoint collection of measurable subsets of a set of finite measure E. If $\phi = \sum_{i=1}^{n} a_i . \chi_{E_i}$, a_i $'s$ are real numbers, $1 \le i \le n$ then prove that $\int_E \phi = \sum_{i=1}^{n} a_i . m(E_i)$.

57) Let f be a bounded function defined on the closed bounded interval $[a, b]$. If f is Riemann integrable over $[a, b]$, then prove that f is Lebesgue integrable over $[a, b]$ and the two integrals are equal.

58) Let f be a bounded measurable function on a set of finite measure E. Then prove that f integrable over E.

59) State and prove Chebychev's inequality.

60) Let f be a non-negative measurable function on E then show that $\int_E f = 0$ if and only if $f = 0$ a.e. on E .

61) Let f be a non-negative measurable function on E. If A and B are disjoint measurable subsets of E then show that $\int_{A \cup B} f = \int_A f + \int_B f$. Also show that if E_0 is a subset of E of measure zero then $\int_E f = \int_{E-E_0} f$.

62) Let f and g be the two non-negative measurable functions. If f is integrable over E and $g(x) < f(x)$ on E then show that g is also integrable and $\int_E f - g = \int_E f - \int_E g$.

63) Let f be a non-negative measurable function which is integrable over E. Show that for given $\epsilon > 0$ $\exists$ $\delta > 0$ such that for every set $A \subseteq E$ with $m(E) < \delta$ we have $\int_E f < \epsilon$.

64) State and prove Beppo Levi's lemma.

65) State and prove Lebesgue Convergence Theorem.

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66) State and prove General Lebesgue Dominated Convergence Theorem.

67) Let f be a bounded function on a set of finite measure E . Prove that f is Lebesgue integrable over E if and only if f is measurable.

68) Prove that a function f is of bounded variation over $[a,b]$ if and only if f is difference of two monotone absolutely continuous real valued functions on $[a,b]$.

69) State and prove Holders inquality.

70) Prove that the L^p space is complete.

71) Prove that a function F is an indefinite integral of some integrable function if and only if F is absolutely continuous on $[a,b]$.

72) State and prove Minkowski inequality for $1 \leq p \leq \infty$.

73) Let f be an increasing function on the closed bounded interval $[a,b]$. Then prove that for each $\alpha > 0 m * \{x \in (a,b) \mid \overline{D}f(x) \geq \alpha\} \leq \frac{1}{\alpha}[f(b) - f(a)]$ and $m * \{x \in (a,b) \mid \overline{D}f(x) = \infty\} = 0.$

74) State and prove Lebesgue Theorem.

75) State and prove Jordan's Theorem.

76) State and prove Young's inequality.

77) Let f be a Lipschitz function on $[a,b]$ show that f is of bounded variation on $[a,b]$.
78) If the function f is Lipschitz on a closed bounded interval $[a,b]$ then show that f is absolutely continuous on $[a,b]$.

79) Let f be the continuous on a closed bounded interval $[a,b]$. Then show that f is absolutely continuous on $[a,b]$ if and only if the family of divided difference functions $\{\mathrm{Diff}_h f\}_{0\leq h\leq 1}$ is uniformly integrable over $[a,b]$.

80) State and prove Fundamental Theorem of Integral Calculus for Lebesgue Integral.

81) Prove that a function f on a closed bounded interval $[a, b]$ is absolutely continuous on$[a, b]$if and only if it is an indefinite integral over $[a, b]$.

82) Let f be integral on a closed bounded interval$[a, b]$. Then prove that $f(x) = 0$ for almost all $x\epsilon[a, b]$ if and only if$\int_{x_1}^{x_2} f = 0$ for all$(x_1, x_2) \subseteq [a, b]$.

83) Let f be integrable over the closed bounded interval$[a, b]$then $\frac{d}{dx}\left[\int_a^x f\right] = f(x)$for almost all $x\epsilon[a, b]$.

84) Define norm on a linear space and prove that $L^1(E)$ is a normed linear space.

85) Define norm on a linear space and prove that $L^\infty(E)$ is a normed linear space.

86) State and prove Cauchy Schwarz inequality.

87) Prove that every convergent sequence in a normed linear space X is Cauchy and Cauchy sequence in X is convergent if it has convergent subsequence.

89) Prove that every rapidly Cauchy sequence in a normed linear space X is Cauchy and every Cauchy sequence in X has rapidly Cauchy subsequence.

90) State and prove Riesz-Fischer theorem.